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ON A PERIODIC MIXED PROBLEM FOR A STRIP*

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The periodic problem of the action of rigid stamps on a strip is considered. The foundations of the stamps are assumed to be arbitrarily convex and symmetric about their vertical axes. The problem is reduced to dual summation equations by a traditional method. Two cases are studied: the corners of the stamp press on the strip (the width of the contact area is known), and the corners of the stamp do not reach the strip (the width of the contact area is unknown). The solution for stamps with flat bases follows as a special case from the solution obtained. This is simultaneously the solution (apart from sign and notation) of a certain doubly-periodic problem for a plane with slits.

The problem under consideration has been studied by other methods in /1–3/.

1. The domain of the strip to be studied lies in the complex $z = x + iy$ plane (see Fig. 1 on which the base of the stamps is shown flat for simplicity). The stamps acting on a strip from both sides have identical width and are arranged symmetrically with period $2b$. Therefore, the problem is periodic, and, consequently, we refer all reasoning to the fundamental period $-b \leqslant x \leqslant b$.

We will write the boundary conditions forthe upper boundary $y = a$

$$v = -f(x), \ |x| < c \qquad (1.1)$$
$$Y_y = 0, \ c < |x| \leqslant b$$
$$X_y = 0, \ |x| \leqslant b$$

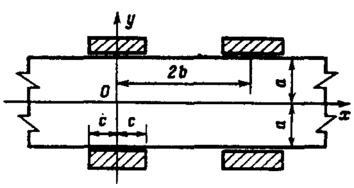

Fig.1

The form of the function $f(x)$ will be indicated below. We denote the pressure under the stamps by $Y_{y*}(x)$ then the load $Y_{ya}(x)$ on the faces of the strip can be represented as follows:

$$Y_{ya}(x) = \begin{cases} Y_{y*}(x), |x| < c \\ 0, \ c < |x| \leqslant b \end{cases}; X_{ya}(x) = 0, \quad |x| \leqslant b \qquad (1.2)$$

We expand the periodic load $Y_{ya}(x)$ in a Fourier series

$$Y_{ya}(x) = \sum_{n=0}^{\infty} a_n \cos \varkappa_n x, \quad \varkappa_n = \frac{n\pi}{b} \qquad (1.3)$$

$$a_0 = \frac{1}{b}\int_0^b Y_{ya}(x)\,dx, \quad a_n = \frac{2}{b}\int_0^b Y_{ya}(x) \cos \varkappa_n r\,dx$$

We now rewrite the boundary conditions (1.1) by using the Kolosov-Muskhelishvili potentials

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$$\varkappa\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)} = 2G(u - if),\ t = x + ia,\ |x| < c \tag{1.4}$$

$$\varphi'(t) + \overline{\varphi'(t)} + t\overline{\varphi''(t)} + \overline{\psi'(t)} = 0,\ t = x + ia,\ c < |x| \leqslant b \tag{1.5}$$

It is sometimes convenient to differentiate condition (1.4) with respect to x

$$\varkappa\varphi'(t) - \overline{\varphi'(t)} - t\overline{\varphi''(t)} - \overline{\psi'(t)} = 2G(u' - if') \tag{1.6}$$
$$t = x + ia,\ |x| < c$$

Assuming the periodic load (1.2) and (1.3) to be given on the faces of the strip, we represent the solution of the problem in the following form:

$$\varphi(z) = \frac{a_0}{4}z + \sum_{n=1}^{\infty}\frac{A_n}{\varkappa_n}\sin\varkappa_n z \tag{1.7}$$

$$\psi(z) = \frac{3}{4}a_0 z + \sum_{n=1}^{\infty}\frac{B_n}{\varkappa_n}\sin\varkappa_n z - z\varphi'(z)$$

$$A_n = a_n\,\mathrm{sh}\,\mu_n/S(\mu_n),\ S(\mu_n) = 2\mu_n + \mathrm{sh}\,2\mu_n$$

$$B_n = a_n(2\mu_n\mathrm{ch}\,\mu_n + \mathrm{sh}\,\mu_n)/S(\mu_n)$$

$$\mu_n = a\varkappa_n = n\pi/\varepsilon_1,\ \varepsilon_1 = b/a$$

In this solution the coefficients a_n are unknown since the pressure under the stamps is unknown. To determine the a_n it is necessary to return to (1.4)–(1.6). We first substitute (1.7) into (1.4) and extract the imaginary part since it is precisely given

$$\frac{a_0}{4} + \sum_{n=1}^{\infty}\frac{a_n}{\mu_n}L(\mu_n)\cos\varkappa_n x = -\frac{2G}{a(\varkappa+1)}f(x) \tag{1.8}$$

$$L(\mu_n) = \mathrm{sh}^2\mu_n/S(\mu_n),\ |x| < c$$

Proceeding in the same was as with (1.7) and (1.6), we arrive at the following equation:

$$\sum_{n=1}^{\infty}a_n L(\mu_n)\sin\varkappa_n x = \frac{2G}{\varkappa+1}f'(x),\ \ |x| < c \tag{1.9}$$

Finally, substituting (1.7) into (1.5), we obtain

$$\sum_{n=0}^{\infty}a_n\cos\varkappa_n x = 0,\ \ c < |x| \leqslant b \tag{1.10}$$

The dual summing equation (1.8) and (1.10) (or (1.9) and (1.10)) is thereby obtained to determine the coefficients a_n. We give the function $f(x)$, which depends on the shape of the stamp base, in the form

$$f(x) = \sum_{k=0}^{\infty}\gamma_k\left(\frac{x}{c}\right)^{2k},\ \ |x| \leqslant c \tag{1.11}$$

The quantity γ_0 is unknown here since it characterizes the rigid displacement of the steamp; the remaining coefficients $\gamma_k\ (k \geqslant 1)$ are given. The stamps are bounded on the sides by the lines $x = \pm c$.

We shall consider two cases separately. The first is when the stamp corners $x = \pm c$ press on the strip. In this case the width of the contact area is known and equals $2c$. In the second case we assume that the stamp corners do not reach the stamp. Here the width of the contact area $2c_1 < 2c$ is unknown, but the solution of the problem will be obtained in a form such that this width must be given in advance, while the other quantities are considered to be unknown.

2. We will examine the first case (the stamp corners press on the strip). The width of the contact area equals the width of the stamps $2c$. We take (1.8) and (1.10) as dual summing equations, taking (1.11) into account

$$\frac{a_0}{4} + \sum_{n=1}^{\infty}\frac{a_n}{\mu_n}L(\mu_n)\cos\varkappa_n x = -\frac{2G}{a(\varkappa+1)}\sum_{k=0}^{\infty}\gamma_k\left(\frac{x}{c}\right)^{2k},\ \ |x| < c \tag{2.1}$$

$$\sum_{n=0}^{\infty}a_n\cos\varkappa_n x = 0,\ \ c < |x| \leqslant b$$

To solve this equation we first construct the following discontinuous function:

$$\frac{\delta_{k0}}{2}+\sum_{n=1}^{\infty}J_{2k}(\varkappa_n c)\cos\varkappa_n x=\begin{cases}(-1)^k\frac{b}{\pi}\frac{T_{2k}\left(\frac{x}{c}\right)}{\sqrt{c^2-x^2}}, & |x|<c\\ 0, & c<|x|\leqslant b\end{cases} \tag{2.2}$$

Here $J_{2k}(t)$ is a Bessel function of the first kind, $T_{2k}(t)$ are Chebyshev polynomials of the first kind, and δ_{k0} is the Kronecker delta.

Now, if the coefficients a_n are taken in the form

$$a_0=-\alpha_0{}^*\frac{P*}{2b}, \quad \varepsilon=\frac{\pi c}{b} \tag{2.3}$$
$$a_n=-\sum_{k=0}^{\infty}(-1)^k\alpha_k{}^*J_{2k}(n\varepsilon), \quad n\geqslant 1$$

the second equation of (2.1) is satisfied automatically by (2.2). Substituting (2.3) into (1.3) and using (2.2) and (1.2), we obtain the pressure under the stamp

$$Y_{ys}(x)=-\frac{P*}{\pi\sqrt{c^2-x^2}}\sum_{k=0}^{\infty}\alpha_k{}^*T_{2k}\left(\frac{x}{c}\right), \quad |x|<c \tag{2.4}$$

($P*$ is the force squeezing against the stamp). Using (2.4) in the equilibrium conditions for the stamp

$$\int_{-c}^{c}Y_{ys}(x)\,dx=-P* \tag{2.5}$$

we find $\alpha_0{}^*=1$.

The formula /4/

$$\cos\varkappa_n x=2\sum_{m=0}^{\infty}(-1)^m\tau_m J_{2m}(n\varepsilon)T_{2m}\left(\frac{x}{c}\right) \tag{2.6}$$
$$\tau_0={}^1/_2, \quad \tau_{m\geqslant1}=1, \quad |x|\leqslant c$$

as well as the following expansion

$$x^k=2^{1-k}\sum_{m=0}^{[k/2]}\nu^{(k)}_{[k/2]-m}C_k{}^mT_{k-2m}(x), \quad |x|\leqslant 1 \tag{2.7}$$
$$\nu_0^{(k)}={}^1/_2, \quad \nu_{n\geqslant1}^{(k)}=1 \quad (k=0,2,4,\ldots), \quad \nu_{n\geqslant0}^{(k)}=1 \quad (k=1,3,5,\ldots)$$

are needed below.

Using (2.7), we express the sum on the right side of the first equation (2.1) in terms of $T_k(x/c)$

$$f(x)=\sum_{k=0}^{\infty}\gamma_k\left(\frac{x}{c}\right)^{2k}=\sum_{s=0}^{\infty}A_s{}^*T_{2s}\left(\frac{x}{c}\right), \quad |x|\leqslant c \tag{2.8}$$
$$A_s{}^*=\sum_{k=s}^{\infty}2^{1-2k}\nu_s^{(2k)}\gamma_kC_{2k}^{k-s}$$

We note that the quantity $A_0{}^*$ is unknown since γ_0 is unknown while the remaining $A_s{}^*$ are known.

To satisfy the first equation in (2.1), we substitute (2.3), (2.6) and (2.8), we interchange the order of summation on the left side and we equate expressions for identical Chebyshev polynomials on the right and left /5/. This results in an infinite system of linear algebraic equations to determine $\alpha_k{}^*$

$$\sum_{k=0}^{\infty}\left(\frac{\pi}{8\varepsilon_1}\delta_{k0}+a_{0k}^*\right)\alpha_k{}^*=2\beta A_0{}^* \tag{2.9}$$
$$\sum_{k=0}^{\infty}a_{mk}^*\alpha_k{}^*=(-1)^m\beta A_m{}^*, \quad m\geqslant 1$$
$$a_{mk}^*=(-1)^{k+m}a_{km}^*=(-1)^k\sum_{n=1}^{\infty}\frac{L(\mu_n)}{n}J_{2k}(n\varepsilon)J_{2m}(n\varepsilon)$$
$$\beta=\frac{\pi}{\varkappa+1}\frac{G}{P*}$$

The width of the stamp $2c$ (and therefore also ε) and the force $P*$ are known in the problem under consideration, but the degree of stamp insertion γ_0 (and therefore alos of $A_0{}^*$) is unknown. Consequently, the right side in the first equation in (2.9) is unknown and the system must be solved twice with two different right sides. The first time, the right side in the first equation in (2.9) must be taken equal to zero, the remaining right sides are retained. The second time, on the other hand, the right side in the first equation must be retained while the right sides in the remaining equations are set equal to zero. Summing the solutions of these two systems we find the solution of system (2.9)

$$\alpha_k^* = \frac{G}{P^*}\frac{c}{\varkappa+1}\left(\tau_k^{(1)} + \tau_k^{(2)}\frac{A_0^*}{c}\right) \quad (k \geqslant 0) \tag{2.10}$$

($\tau_k^{(1)}$ and $\tau_k^{(2)}$ are numerical coefficients). The A_0^* in the solution obtained is unknown. To determine it we use the fact that $\alpha_0^* = 1$. Hence, and from (2.10) for $k = 0$ we express A_0^* in terms of P^*. The dependence between the degree of stamp insertion γ_0 and the force P^* is thereby also determined. We note that in the case of a plane stamp $\gamma_k = 0$ $(k \geqslant 1)$, consequently (see (2.8)) $A_0^* = \gamma_0$. Investigation of system (2.9) shows that, depending on the values of the parameters ε and ε_1 it is quasiregular in the worst case.

3. Now we examine the second case (the stamp corners do not reach the strip). We here assume the width $2c_1$ of the contact area to be given, and the pressure under the stamp, the degree of stamp insertion, and the force P^{**} acting on the stamp to be desired.

We take (1.9) and (1.10) as the dual equation. We write the derivative in (1.9) in the following form

$$f'(x) = \frac{2}{c_1}\sum_{k=1}^{\infty} k\gamma_k^*\left(\frac{x}{c_1}\right)^{2k-1}, \quad |x| \leqslant c_1; \quad \gamma_k^* = \gamma_k\left(\frac{c_1}{c}\right)^{2k} \quad (k \geqslant 0) \tag{3.1}$$

The following discontinuous function

$$\frac{\varepsilon_2}{4}\delta_{k0} + \sum_{n=1}^{\infty}\frac{J_{2k+1}(n\varepsilon_2)}{n}\cos\varkappa_n x = \begin{cases} \dfrac{(-1)^k}{2k+1}\sqrt{1-\dfrac{x_t}{c_1^2}}\,U_{2k}\left(\dfrac{x}{c_1}\right), & |x| \leqslant c_1 \\ 0, \quad c_1 \leqslant |x| \leqslant b; & \varepsilon_2 = \pi c_1/b \end{cases} \tag{3.2}$$

is needed to solve the dual equation.

Here $U_{2k}(t)$ are Chebyshev polynomials of the second kind.

We take the solution of (1.9) and (1.10) in the form:

$$a_0 = -\frac{P^{**}}{2b}\alpha_0^{**}$$
$$a_n = -\frac{2P^{**}}{n\pi c_1}\sum_{k=0}^{\infty}(-1)^k(2k+1)\alpha_k^{**}J_{2k+1}(n\varepsilon_2) \quad (n \geqslant 1) \tag{3.3}$$

As follows from (3.2), equation (1.10) will be satisfied automatically here. Substituting (3.3) into (1.3) and using (1.2), we obtain the pressure under the stamps (P^{**} is the force acting on the stamp)

$$Y_{y_s}(x) = -\frac{2P^{**}}{\pi c_1}\sqrt{1-\frac{x^2}{c_1^2}}\sum_{k=0}^{\infty}\alpha_k^{**}U_{2k}\left(\frac{x}{c_1}\right), \quad |x| \leqslant c_1 \tag{3.4}$$

Substituting (3.4) into the stamp equilibrium condition analogous to (2.5) we obtain that $\alpha_0^{**} = 1$.

To satisfy equation (1.9), we represent $\sin\varkappa_n x$ as follows /4/:

$$\sin\varkappa_n x = 2\sum_{m=0}^{\infty}(-1)^m J_{2m+1}(n\varepsilon_2)\,T_{2m+1}\left(\frac{x}{c_1}\right), \quad |x| \leqslant c_1 \tag{3.5}$$

We also express the right side of (1.9), i.e., (3.1), in terms of $T_k(t)$ by using (2.7)

$$f'(x) = \sum_{m=0}^{\infty} B_m^* T_{2m+1}\left(\frac{x}{c_1}\right), \quad |x| \leqslant c_1$$
$$B_m^* = \sum_{k=m+1}^{\infty} 2^{2-2k}k\gamma_k^* C_{2k-1}^{k-m-1} \tag{3.6}$$

Now substituting (3.3), (3.5) and (3.6) into (1.9), changing the order of summation on the left and equating right and left expressions for identical Chebyshev polynomials of the first kind, we obtain an infinite system of linear algebraic equations to determine α_k^{**} $(k \geqslant 0)$

$$\sum_{k=0}^{\infty} a_{mk}^{**}\alpha_k^{**} = (-1)^{m+1}\beta^* B_m^* \quad (m \geqslant 0), \quad \beta^* = \frac{\pi}{\varkappa+1}\frac{G}{P^{**}}$$
$$a_{mk}^{**} = (-1)^{k+m}\frac{2k+1}{2m+1}a_{km}^{**} =$$
$$(-1)^k(2k+1)\sum_{n=1}^{\infty}\frac{L(\mu_n)}{n}J_{2k+1}(n\varepsilon_2)\,J_{2m+1}(n\varepsilon_2) \tag{3.7}$$

Investigation of system (3.7) shows that, at the least it is quasiregular depending on the values of the parameters ε_1 and ε_2. The solution of this system can be written thus (τ_k^* are numerical coefficients):

$$\alpha_k^{**} = \tau_k^* \frac{c_1}{\varkappa + 1} \frac{G}{P^{**}} \tag{3.8}$$

From (3.8) for $k = 0$ and the fact that $\alpha_0^{**} = 1$, we determine the force P^{**}.

To determine the degree of stamp insertion $\gamma_0 = \gamma_0^*$ we turn to (1.8), where $f(x)$ (see (1.11)) is written thus (we use (2.7)):

$$f(x) = \sum_{k=0}^{\infty} \gamma_k^* \left(\frac{x}{c_1}\right)^{2k} = \sum_{m=0}^{\infty} A_m^{**} T_{2m}\left(\frac{x}{c_1}\right)$$

$$A_m^{**} = \sum_{k=m}^{\infty} 2^{1-2k} \nu_m^{(2k)} \gamma_k^* C_{2k}^{k-m}, \quad |x| \leqslant c_1$$

Furthermore, we use (2.6) in which the ε must be replaced by ε_2, c by c_1, and $J_{2m}(n\varepsilon_2)$ must be represented by the following sum /6/:

$$J_{2m}(n\varepsilon_2) = (n\varepsilon_2/(4m))\,[J_{2m-1}(n\varepsilon_2) + J_{2m+1}(n\varepsilon_2)]$$

Substituting all this into (1.8) and proceeding as in the derivation of system (3.7), we obtain an infinite system of equations, in which all the equations except the first are satisfied identically by virtue of (3.7). The first equation appears as follows:

$$\frac{\pi}{16}\alpha_0^{**} + \sum_{k=0}^{\infty} (-1)^k (2k+1) \lambda_k \alpha_k^{**} = \varepsilon_1 \beta^* A_0^{**}$$

$$\lambda_k = \sum_{n=1}^{\infty} \frac{L(\mu_n)}{n^2} J_0(n\varepsilon_2) J_{2k+1}(n\varepsilon_2)$$

We determine A_0^{**} from this equation (γ_0^* occurs it) since we determined the force P^{**} earlier.

By virtue of the symmetry, the tangential stress on the strip axis is $X_y = 0$ and the vertical displacement is $v = 0$, hence, the solution obtained is evidently a solution of the problem for a strip of width a, resting without friction on an absolutely rigid base and loaded along the upper face by stamps /1/.

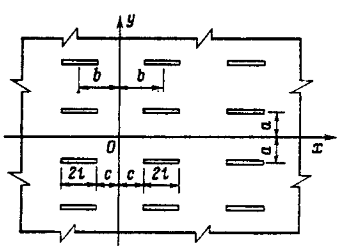

Fig.2

We consider the following problem. A plane is weakened by a doubly-periodic system of slits (Fig.2) and is stretched at infinity in the Oy direction by a mean stress $Y_y = q$. Any line $y = ka$ ($k = \pm 1, \pm 3, \pm 5, \ldots$) as well as $y = na$ ($n = 0, \pm 2, \pm 4, \ldots$) can be taken as the axis of symmetry; hence it follows that the ligaments between the slits always remain rectilinear.

We extract a strip $|x| \leqslant \infty$, $|y| \leqslant a$ from the plane under consideration and we write the following boundary conditions for the fundamental period of this strip

$$v(x) = \pm f_0 = \text{const}, \ |x| < c, \ y = \pm a$$
$$Y_y(x) = 0, \ c < |x| \leqslant b, \ y = \pm a; \ X_y(x) = 0, \ |x| \leqslant b, \ y = \pm a$$

These boundary conditions are a special case of condition (1.1). This problem can therefore be regarded as solved (see Sect.2). In particular, it is here necessary to set $P^* = -2qb$. Then the stress Y_y in the ligament $|x| < c$, $y = a$ will be determined by (2.4) with the mentioned replacement for the force P^*. The following stress intensity coefficient follows from this formula

$$K_1 = \frac{2qb}{\sqrt{\pi c}} \sum_{n=0}^{\infty} \alpha_n^*$$

Here $2c$ is the width of the ligament (the width of the slit is $2l = 2(b - c)$). The system of slits (cracks) is non-equilibrium in the case under consideration.

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